Solution

① Write the equation expressing the global neutrality of the medium:

↳ The total charge in the medium is zero $\Leftrightarrow$ $\boxed{Q_{tot} = \int_{\mathcal{V}} \rho_{in}\, d\mathcal{V} = 0}$

② We will evaluate the projection of $\vec{\mathcal{P}}$ along $\vec{e}_x$ (same can be done for x, y)

$$\vec{\mathcal{P}} = \int_{\mathcal{V}} \vec{r} \rho_{in}(\vec{r})\, d\mathcal{V} \Rightarrow \mathcal{P}_x = \int_{\mathcal{V}} x \rho_{in}(\vec{r})\, d\mathcal{V} = -\int_{\mathcal{V}} x \operatorname{div}\vec{P}(\vec{r})\, d\mathcal{V}$$

But $\operatorname{div}(x\vec{P}) = x \operatorname{div}(\vec{P}) + \vec{P}.\overrightarrow{\operatorname{grad}}(x)$ and $\overrightarrow{\operatorname{grad}}(x) = \vec{e}_x$, so:

$$\mathcal{P}_x = \int_{\mathcal{V}} P_x(\vec{r})\, d\mathcal{V} - \int_{\mathcal{V}} \operatorname{div}(x\vec{P})(\vec{r})\, d\mathcal{V} \quad \text{with } P_x = \vec{P}.\vec{e}_x$$

Now, since $\vec{P}(\vec{r}) = 0$ for $\vec{r} \notin \mathcal{V}$, by continuity $\vec{P} = \vec{0}$ on the boundary surface S. By integrating over $\mathcal{V}$:

$$\mathcal{P}_x = \int_{\mathcal{V}} P_x(\vec{r})\, d\mathcal{V} - \oint_S x\vec{P}.\vec{n}\, dS$$

$$= \int_{\mathcal{V}} P_x(\vec{r})\, d\mathcal{V} \quad \text{because } \vec{P} = \vec{0} \text{ on } S$$

Since we can repeat for y and z components we proved that

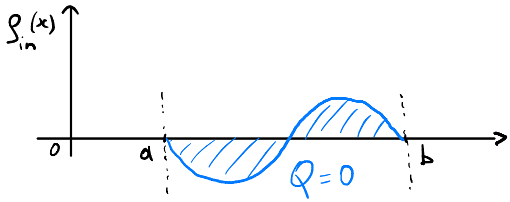
$\boxed{\vec{\mathcal{P}} = \int_{\mathcal{V}} \vec{P}\, d\mathcal{V}}$ → $\vec{P}(\vec{r})$ plays the role of a <u>volumic density of dipole moment</u> of the charge distrib.

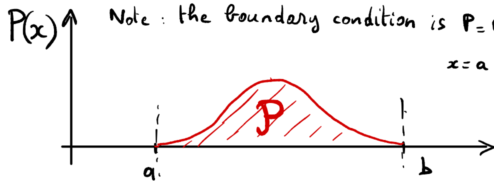
③ * $\boxed{\operatorname{div}\vec{P} \equiv -\rho_{in} = -\varepsilon_0 \operatorname{div}\vec{E}}$ from Maxwell equation

* However, outside $\mathcal{V}$ $\vec{P} = \vec{0}$ while $\vec{E} \neq \vec{0}$, so $\boxed{\vec{P} \neq -\varepsilon_0 \vec{E}}$

④ $\left.\begin{array}{l}\rho_{in} \to [C.m^{-3}] \\ \operatorname{div} \to [m^{-1}]\end{array}\right\}$ $\boxed{\vec{P} \text{ has unit of } [C.m^{-2}]}$, which is indeed $\dfrac{\text{dipole moment}}{\text{volume}} = \dfrac{C.m}{m^3}$

⑤





Note : the boundary condition is P = 0 at x = a & b

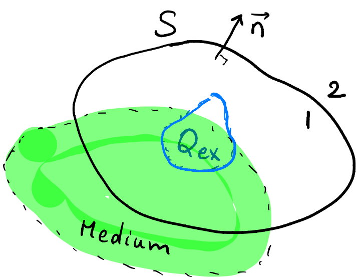
⑥ With both internal and external charges, Maxwell - Gauss equation is $\mathrm{div}(\varepsilon_0 \vec{E}) = \rho_{in} + \rho_{ex}$, and introducing the field $\vec{P}(\vec{r})$:

$$\mathrm{div}(\varepsilon_0 \vec{E}) = -\mathrm{div}\vec{P} + \rho_{ex} \quad \Longrightarrow \quad \mathrm{div}(\varepsilon_0 \vec{E} + \vec{P}) = \rho_{ex}$$

We therefore define a new field $\boxed{\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}}$ satisfying $\boxed{\mathrm{div}\vec{D} = \rho_{ex}}$

$\vec{D}$ is often called "displacement" field or "electric field density"

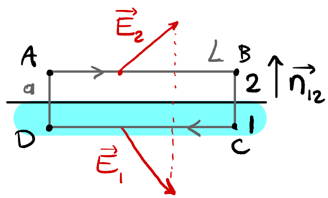
It has dimensions $[C.m^{-2}]$ like $\vec{P}$.



N.B. : We can generalize Gauss theorem in a medium with external charge Q_{ex} :

$$\int_V \mathrm{div}\vec{D}\, dV = \int_V \rho_{ex}\, dV \Rightarrow \boxed{\oint_S \vec{D}.\vec{n}\, dS = Q_{ex}}$$

⑦ Interface condition for $\vec{E}$

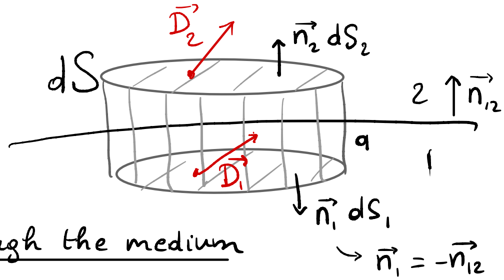


We consider a closed curve encircling the interface with $a \ll L$. Because $\vec{E} = -\vec{\mathrm{grad}}\, V$ the integral of $\vec{E}$ over this contour is zero. As $\frac{a}{L} \to 0$ we get :

$$0 = \vec{E}_1 . \vec{AB} + \vec{E}_2 . \vec{CD} = L(E_{1,t} - E_{2,t})$$

meaning that the tangential component of the electric field is always continuous at the interface, which we can write also as : $\boxed{\vec{n}_{12} \times (\vec{E}_2 - \vec{E}_1) = \vec{0}}$ (i.e., $\vec{E}_2 - \vec{E}_1 \parallel \vec{n}_{12}$)

* Interface condition for $\vec{D}$



We have seen that without external charges present : $\oint_S \vec{D}.\vec{n}\, dS = 0$
for any closed surface , <u>even if it cuts through the medium</u>

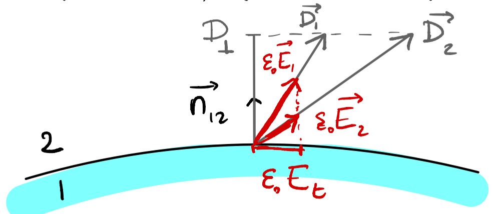
As the height of the cylinders goes to zero :

$$\oint_S \vec{D}.\vec{n}\, dS = \vec{D}_2 \cdot \vec{n}_{12}\, dS_2 - \vec{D}_1 \cdot \vec{n}_{12}\, dS_1 = \vec{n}_{12} \cdot (\vec{D}_2 - \vec{D}_1)\, dS = 0$$

8 If there is an external surface charge density : $Q_{ex} = \sigma_{ex}\, dS$
then $\boxed{\vec{n}_{12} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma_{ex}}$

<u>Remark</u> : Contrary to $\vec{E}$, $\vec{D}$ is not the gradient of a potential, and as such in general the tangential component of $\vec{D}$ <u>is discontinuous</u> at an interface .
But when only ρ_{in} is present (usual case), then the <u>normal component is continuous</u> .

Graphically , for an <u>isotropic medium</u> (as considered everywhere here)



9 <u>Potential created by the object</u>

$$V_{el}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \frac{-\mathrm{div}\, \vec{P}(\vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|}\, d\mathcal{V}$$

Now we use $\mathrm{div}(f\vec{P}) = f\, \mathrm{div}\, \vec{P} + \vec{P}.\, \overrightarrow{\mathrm{grad}}\, f$
with $f(\vec{r}\,') = \dfrac{1}{\|\vec{r} - \vec{r}\,'\|}$

i.e. $$\frac{\mathrm{div}\, \vec{P}}{\|\vec{r} - \vec{r}\,'\|} = \mathrm{div}\left(\frac{\vec{P}}{\|\vec{r} - \vec{r}\,'\|}\right) - \vec{P} \cdot \frac{(\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3}$$

The first term is zero : $\int_{\mathcal{V}} -\operatorname{div}\left(\frac{\vec{P}}{\|\vec{r}-\vec{r}'\|}\right) d\mathcal{V} = \oint_S \frac{\vec{P}.\vec{n}}{\|\vec{r}-\vec{r}'\|} dS = 0$

as $\vec{P} = \vec{0}$ on the boundary S (or we could take S just outside the object)

It remains : $$\boxed{V_{ee}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \vec{P}(\vec{r}') \cdot \frac{\vec{r}-\vec{r}'}{\|\vec{r}-\vec{r}'\|^3} d\mathcal{V}}$$

The potential is identical to that of a volumic distribution of electric dipoles with elementary dipole moments $\vec{P}(\vec{r})\, d\mathcal{V}$

This equivalence is well suited to dielectric media with bound charges. For conductors it is more formal since there are no localised dipoles, yet mathematically it remains correct.

Summary : The state of polarisation of a medium can be described equivalently
* by the distribution of internal charge density $\rho_{in}(\vec{r})$
* by the polarisation density field $\vec{P}(\vec{r})$

When the potential is measured far away from the medium :

$$\boxed{V_{ee}(\vec{r}) \simeq \frac{1}{4\pi\varepsilon_0} \frac{\vec{\mathcal{P}}.\vec{r}}{r^3}}$$

(we assumed $r \gg r'$ with the origin of coordinates inside the medium)